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A Generalized FKKM Theorem and Variational Inequality

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Abstract

We present a generalized FKKM Theorem and it's application to the existence of solution for the variationals inequalities using a generalized coercivity type condition for correspondences defined in L-space.

Key words and phrases: L-structures, L-spaces, L-KKM correspondences, L-coercing family and variational inequality.
Classification-JEL: C02, C69, C72.

The purpose of this article is to give a generalization of FKKM Theorem [KKM] and it's application in variational inequalities. All these results extend classical results obtained in topological vector spaces by Fan [F1] [F2], Dugundji and Granas [DG], Ding and Tan [DT] and Yen [Y] as well as results obtained in H-spaces by Bardaro and Ceppitelli [BC1], [BC2], in convex spaces in the sense of Lassonde [L] and in L-spaces by Chebbi, Gourdel and Hammami [CGH], [GH].

In this paper, we will use the same notation as in [CGH]. We remind the definition given in [CGH] of L-KKM correspondences, which extend the notion of KKM correspondences to L-spaces, and the concept of L-coercing family for correspondences defined in L-spaces. Let A be a subset of a vector space X. We denote by $\langle A\rangle$ the family of all nonempty finite subsets of A and $convA$ the convex hull of A. Since topological spaces in this paper are not supposed to be Hausdorff, following the terminology used in [B], a set is called *quasi-compact* if it satisfies the Finite Intersection Property while a Hausdorff quasi-compact is called compact. In what follows, the correspondences are represented by capital letters F, G, Q, S, Γ, ..., and the single valued functions will be represented by small letters. We denote by $graphF$ the graph of the correspondence F. If X and Y are two topological spaces, $\zeta(X,Y)$ denotes the set of all continuous functions from X to Y.

If n is any integer, Δ_n denotes the unit-simplex of $\mathbb{R}^{n+1}$ and for every $J \subset \{0,1,\dots,n\}$, Δ_J denotes the face of Δ_n corresponding to J. Let X be

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a topological space. An *L-structure* (also called *L-convexity*) on X is given by a correspondence $\Gamma : \langle X \rangle \to X$ with nonempty valued such that for every $A = \{x_0, ..., x_n\} \in \langle X \rangle$, there exists a continuous function $f^A : \Delta_n \to \Gamma(A)$ such that for all $J \subset \{0, \ldots, n\}$, $f^A(\Delta_J) \subset \Gamma(\{x_j, j \in J\})$. Such a pair (X, Γ) is called an *L-space*. A subset $C \subset X$ is said to be *L-convex* if for every $A \in \langle C \rangle$, $\Gamma(A) \subset C$. A subset $P \subset X$ is said to be *L-quasi-compact* if for every $A \in \langle X \rangle$, there exists a quasi-compact L-convex set D such that $A \cup P \subset D$. Clearly, if C is an L-convex subset of an L-space (X, Γ), then the pair $(C, \Gamma_{|\langle C \rangle})$ is an L-space.

1 A Generalized FKKM Theorem

In this section, we first remind some known definitions of L-KKM correspondences and L-coercing family quoted in [CGH], then we give a generalized FKKM Theorem and we deduce a more adapted theorem to study the variational inequality.

Definition 1.1 *Let (X, Γ) be an L-space and $Z \subset X$ an arbitrary subset. A correspondence $F : Z \to X$ is called L-KKM if and only if:*

$$\forall A \in \langle Z \rangle, \qquad \Gamma(A) \subset \bigcup_{x \in A} F(x).$$

Definition 1.2 *Let Z be an arbitrary set of an L-space (X, Γ), Y a topological space and $s \in \zeta(X, Y)$. A family $\{(C_a, K)\}_{a \in X}$ is said to be L-coercing for a correspondence $F : Z \to Y$ with respect to s if and only if:*

(i) K is a quasi-compact subset of Y,

(ii) for each $A \in \langle Z \rangle$, there exists a quasi-compact L-convex set D^A in X containing A such that:

$$x \in D^A \Rightarrow C_x \cap Z \subset D^A \cap Z,$$

(iii) $$\left\{ y \in Y \mid y \in \bigcup_{z \in s^{-1}(y)} \bigcap_{x \in C_z \cap Z} F(x) \right\} \subset K.$$

For more explanation of the L-coercivity and to see that this coercivity can't be compared to the coercivity in the sense of Ben-El-Mechaiekh, Chebbi and Florenzano in [BCF], see [CGH].

Definition 1.3 *If X is a topological space, a subset B of X is called strongly compactly closed (open respectively) if for every quasi-compact subset K of X, $B \cap K$ is closed (open, respectively) in K.*

We remind the following result given in [CGH], which is an extension of a lemma in [F1] to L-spaces.

Lemma 1.1 *Let (X, Γ) be an L-space, Z a nonempty subset of X and $F : Z \to X$ an L-KKM correspondence with strongly compactly closed values. Suppose that for some $z \in Z$, the correspondence $F(z)$ is quasi-compact, then $\bigcap_{x \in Z} F(x) \neq \emptyset$.*

Proof: see [CGH].

The main result of this paper is the following generalized FKKM Theorem (see for example Theorem 4 in [F2] and Theorem 1 in [CGH]:

Theorem 1.1 *Let Z be an arbitrary set in the L-space (X, Γ), Y an arbitrary topological space and F,$G : Z \to Y$ two correspondences such that:*

(a) for every $x \in Z$, $F(x)$ is strongly compactly closed,

(b) for every $x \in Z$, $G(x) \subset F(x)$,

(c) there is a function $s \in \zeta(X, Y)$ satisfying :

1. *the correspondence $R : Z \to X$ defined by $R(x) = s^{-1}(F(x))$ is L-KKM,*
2. *there exists an L-coercing family $\{(C_a, K)\}_{a \in X}$ for G with respect to s,*
3. *for each quasi-compact L-convex set C in X:*

$$\bigcap_{x \in C \cap Z} G(x) \cap s(C) \neq \emptyset \Leftrightarrow \bigcap_{x \in C \cap Z} F(x) \cap s(C) \neq \emptyset.$$

Then $\bigcap_{x \in Z} F(x) \neq \emptyset$ more precisely $K \bigcap \left(\bigcap_{x \in Z} F(x) \right) \neq \emptyset$.

Proof: The correspondence F has strongly compactly closed values, then in order to prove that:

$$K \bigcap \left(\bigcap_{x \in Z} F(x) \right) \neq \emptyset,$$

it suffices to prove that for each finite subset A of Z, $\left(\bigcap_{x\in A} F(x)\right) \cap K \neq \emptyset$.

Let $A \in \langle Z\rangle$, by condition (ii) of Definition 1.2, there exists a quasi-compact L-convex set D^A containing A such that for all $y \in D^A$, $C_y \cap Z \subset D^A \cap Z$. Consider now the correspondence $R^A : D^A \cap Z \to D^A$ defined by $R^A(x) = R(x) \cap D^A$. By Hypothesis $(c.1)$ and the L-convexity of D^A, it is immediate that the correspondence R^A is L-KKM. Next, by the continuity of s, $F(x) \cap s(D^A)$ is closed in $s(D^A)$ then $R^A(x) = s_0^{-1}(F(x) \cap s(D^A))$, where s_0 is the restriction of s to D^A, is closed in D^A and consequently $R^A(x)$ is quasicompact. Since $(D^A, \Gamma_{|\langle D^A\rangle})$ is also an L-space, we deduce by Lemma 1.1 that $\bigcap_{x\in D^A} R^A(x) \neq \emptyset$, then $\bigcap_{x\in D^A\cap Z} R^A(x) \neq \emptyset$. Since for all $x \in D^A \cap Z$, $s(R^A(x)) \subset F(x) \cap s(D^A)$, we have: $\bigcap_{x\in D^A\cap Z} \big(F(x) \cap s(D^A)\big) \neq \emptyset$ then by $(c.3)$, $\bigcap_{x\in D^A\cap Z} \big(G(x) \cap s(D^A)\big) \neq \emptyset$. To finish the proof, we will show that:
$\bigcap_{x\in D^A\cap Z} \big(G(x) \cap s(D^A)\big) \subset \bigcap_{x\in A} F(x) \cap K$.
Indeed, it is clear by (b) that $\bigcap_{x\in D^A\cap Z} \big(G(x) \cap s(D^A)\big) \subset \bigcap_{x\in A} F(x)$, then it only remains to show that: $\bigcap_{x\in D^A\cap Z} \big(G(x) \cap s(D^A)\big) \subset K$. Let $y \in \bigcap_{x\in D^A\cap Z} \big(G(x) \cap s(D^A)\big)$, then $y \in s(D^A)$ which implies that there exists $z \in s^{-1}(y) \cap D^A$. By condition (ii) of Definition 1.2, $C_z \cap Z \subset D^A \cap Z$, it follows that $y \in \bigcup_{z\in s^{-1}(y)} \bigcap_{x\in C_z\cap Z} G(x)$. Hence, by hypothesis $(c.2)$, $y \in K$ and the theorem is proved. ■

Remark 1.1 *(1) The main result of [CGH] (Theorem 1) becomes an immediate corollary of Theorem 1.1: it suffices to take* $F = G$.

(2) In view of our approach, it is possible to state a weakened version of Theorem 1 in [CGH] by replacing the coercivity on F *by a coercivity on* G *together with condition* $(c.3)$ *of our Theorem 1.1.*

Corollary 1.1 *Under the conditions of Theorem 1.1, if we assume in addition that* X *is a quasi-compact set and* s *is a surjective function, then we can reinforce the conclusion:*

$$\bigcap_{x\in Z} G(x) \neq \emptyset.$$

Proof: All the requirement of Theorem 1.1 are satisfied then $\bigcap_{x\in Z} F(x) \neq \emptyset$. By the definition of L-space, it is clear that X is an L-convex set. In addition, X is a quasi-compact set and $s(X) = Y$, then by assumption $(c.3)$, $\bigcap_{x\in Z} G(x) \neq \emptyset$. ■

Remark 1.2 *It is obvious as in [DG] that if we add the following condition:*

$$\bigcap_{x\in Z} F(x) \neq \emptyset \Leftrightarrow \bigcap_{x\in Z} G(x) \neq \emptyset$$

in Theorem 1.1 then in addition to $\bigcap_{x\in Z} F(x) \neq \emptyset$ *we have* $\bigcap_{x\in Z} G(x) \neq \emptyset$.

The next theorem is more specially adapted to the study of variational inequality. It can be seen as a corollary of Theorem 1.1 and it is a generalization of Theorem II [L] and Corollary 1.4 [DG].

Theorem 1.2 *Let Z be an arbitrary set in the L-space (X, Γ), Y an arbitrary topological space and F, $G : Z \to Y$ two correspondences such that:*

(a) for every $x \in Z$, $F(x)$ is strongly compactly closed,

(b) for every $x \in Z$, $G(x) \subset F(x)$,

(c) there is a surjective function $s \in \zeta(X, Y)$ satisfying :

1. *he correspondence $R : Z \to X$ defined by $R(x) = s^{-1}(F(x))$ is L-KKM,*
2. *there exists an L-coercing family $\{(C_a, K)\}_{a\in X}$ for G with respect to s,*
3. *for each L-convex set C in X:*

$$\bigcap_{x\in C\cap Z} G(x) \cap s(C) \neq \emptyset \Leftrightarrow \bigcap_{x\in C\cap Z} F(x) \cap s(C) \neq \emptyset.$$

Then $\bigcap_{x\in Z} G(x) \neq \emptyset$.

Proof: It is obvious to see that Assumption $(c.3)$ of Theorem 1.2 imply Assumption $(c.3)$ of Theorem 1.1, then $\bigcap_{x\in Z} F(x) \neq \emptyset$. By the definition of L-space, it is clear that X is an L-convex set and hence, for the particular case where $C = X$, Assumption $(c.3)$ implies that $\bigcap_{x\in Z} G(x) \neq \emptyset$ and the theorem is proved. ■

2 Application to variational inequalities

In this section we will prove the existence of solutions of variational inequalities using Theorem 1.2.

Let E and P denote two real topological vector space, X a nonempty convex set in E and $\langle \cdot, \cdot \rangle$ a bilinear form on $P \times E$ whose for each fixed $v \in P$, the restriction of $\langle v, \cdot \rangle$ on any quasi-compact subset Q of X is continuous[2] (the natural example is between a normed topological vector space E and its dual space equipped with the strong topology).

Definition 2.4 *A non empty valued correspondence $T : X \to P$ is said to be monotone if for each (x, u) and (y, v) in the graph of T, $\langle u - v, x - y \rangle \geq 0$.*

Remark 2.3 *One checks easily that if a correspondence T is upper hemi-continuous in the sense of Cornet [C1] (see for example [C2] and [F]) then the following condition used by Lassonde [L] for monotone correspondences is satisfied[3]:*
For any $(x, y) \in X \times X$, the function $h_{xy} : [0, 1] \to \mathbb{R}$ defined, for all $t \in [0, 1]$, by: $h_{xy}(t) = \inf_{u \in T((1-t)y + tx)} \langle u, y - x \rangle$ is lower semi-continuous at point $t = 0$, (resp. the function $\tilde{h}_{xy} : [0, 1] \to \mathbb{R}$ defined by for all $t \in [0, 1]$, $\tilde{h}_{xy}(t) = \sup_{u \in T((1-t)y + tx)} \langle u, x - y \rangle$ is upper semi-continuous at point $t = 0$).

The following theorem is a general version of one of the basic facts in the theory of variational inequalities (see for example [HS], [DG] and [L]).

Theorem 2.3 *Let $T : X \to P$ be a non empty monotone correspondence, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ a convex function lower semi-continuous on any quasi-compact subset of X[4]. Let us suppose that there exists a family $\{(C_x, K)\}_{x \in X}$ of pairs of sets satisfying:*

(a) K is a quasi-compact subset of X,

(b) for each $A \in \langle X \rangle$, there exists a quasi-compact convex set D^A containing A such that:

$$x \in D^A \Rightarrow C_x \subset D^A,$$

[2]Which is equivalent, if we suppose that for all $x \in Z$, $\varphi_v(x) = \langle v, x \rangle$, to : for every closed subset F of $\mathbb{R}$, $\varphi^{-1}(F)$ is a strongly compactly closed subset.

[3]It suffices to consider p equal to the (continuous) linear form $\langle \cdot, y - x \rangle$ in the following definition given by Cornet: a correspondence $F : X \to P$ is said upper hemi-continuous in a point $x_0 \in X$ in the sense of Cornet if for any continuous linear function p, the function $h : x \to \sup_{y \in \varphi(x)} p(y)$ (resp. $\tilde{h} : x \to \inf_{y \in \varphi(x)} p(y)$) is upper semi-continuous (resp. lower semi-continuous) at the point x_0.

[4]Or equivalently: for every $\alpha \in \mathbb{R}$, $\varphi^{-1}(] - \infty, \alpha])$ is a strongly compactly closed set.

(c) $\left\{ y \in X \mid \varphi(y) \leq \varphi(x) + \sup_{v \in T(y)} \langle v, x-y \rangle \text{ for all } x \in C_y \right\} \subset K,$

(d) *for each* $(x,y) \in X \times X$*, the function* $h_{xy} : [0,1] \to \mathbb{R}$ *given for* $t \in [0,1]$ *by* $h_{xy}(t) = \sup_{u \in T((1-t)y+tx)} \langle u, x-y \rangle$ *is upper semi-continuous at* $t = 0$.

Then there is a point $y_0 \in X$ *such that,*

$$\varphi(y_0) \leq \varphi(x) + \sup_{v \in T(y_0)} \langle v, x - y_0 \rangle \qquad \forall x \in X.$$

Proof: The proof is similar to the proof of [DG] and [L]. For each $x \in X$, let

$$G(x) = \{ y \in X \mid \varphi(y) \leq \varphi(x) + \sup_{v \in T(y)} \langle v, x-y \rangle \},$$

we have to show that Theorem 1.2 can be applied in order to get $\bigcap_{x \in X} G(x) \neq \emptyset$. Let us now consider the correspondence

$$F(x) = \{ y \in X \mid \varphi(x) \geq \varphi(y) + \sup_{u \in T(x)} \langle u, y-x \rangle \}.$$

We will verify that G and F satisfies requirements of Theorem 1.2 (with $Z = X$ and $s =$ the identity function).

(i.1) From the l.s.c assumption of φ and the "regularity" assumption of the bilinear form $\langle u, . \rangle$, it follows that $F(x)$ is strongly compactly closed in X for each $x \in X$.

(i.2) Let us prove that for every $x \in X$, $G(x) \subset F(x)$: Let $y \in G(x)$, then $\varphi(y) \leq \varphi(x) + \sup_{v \in T(y)} \langle v, x-y \rangle$. By the monotonicity of T, we have: for all $u \in T(x)$ and $v \in T(y)$, $\langle u, x-y \rangle \geq \langle v, x-y \rangle$ then

$$\inf_{u \in T(x)} \langle u, x-y \rangle \geq \sup_{v \in T(y)} \langle v, x-y \rangle$$

consequently

$$- \sup_{u \in T(x)} \langle u, y-x \rangle \geq \sup_{v \in T(y)} \langle v, x-y \rangle$$

which implies $\sup_{u \in T(x)} \langle u, y-x \rangle + \varphi(y) \leq \varphi(x)$, i.e. $y \in F(x)$.

(ii.1) We will prove that F is KKM. Let $y \in conv\{x_1, \ldots, x_n\}$, then there exists $\alpha_i \in [0,1]$ for $i = 1, \ldots, n$ such that $y = \Sigma_{i=1}^n \alpha_i x_i$ and $\Sigma_{i=1}^n \alpha_i =$

1. By the monotonicity of T, for all $u \in T(x_i)$ and $v \in T(y)$, $\langle u, x_i - y\rangle \geq \langle v, x_i - y\rangle$, then

$$\inf_{u\in T(x_i)} \langle u, x_i - y\rangle \geq \langle v, x_i - y\rangle$$

consequently

$$\sup_{u\in T(x_i)} \langle u, y - x_i\rangle \leq -\langle v, x_i - y\rangle$$

which implies $\Sigma_{i=1}^n \alpha_i \sup\limits_{u\in T(x_i)} \langle u, y-x_i\rangle \leq 0$. It follows from the convexity of φ that $\varphi(y) \leq \Sigma_{i=1}^n \alpha_i \varphi(x_i)$. The two previous inequalities allows us to deduce that: $\Sigma_{i=1}^n \alpha_i \left(\sup\limits_{u\in T(x_i)} \langle u, y - x_i\rangle + \varphi(y) - \varphi(x_i)\right) \leq 0$. Therefore, there exists $i \in \{1, \ldots, n\}$ such that $\sup\limits_{u\in T(x_i)} \langle u, y - x_i\rangle + \varphi(y) \leq \varphi(x_i)$, then $y \in \bigcup\limits_{i=1}^{n} F(x_i)$ and F is KKM.

(ii.2) The assumptions (a), (b) and (c), mean exactly that $\{(C_x, K)\}_{x\in X}$ is a coercing family of the correspondence G.

(b.3) Let C be any non-empty convex subset of X. Due to the inclusion between F and G, it is enough to show $\bigcap\limits_{x\in C} (F(x) \cap C) \subset \bigcap\limits_{x\in C} (G(x) \cap C)$. Let $y \in \bigcap\limits_{x\in C} (F(x) \cap C)$. Let us fix z in C and prove that $y \in G(z)$. Obviously, we may assume $\varphi(z) < +\infty$. Since $y \in F(z)$, this implies that $\varphi(y)$ is also finite. For each $0 < t < 1$, let $z_t = (1-t)y + tz$. Since C is convex, then $z_t \in C$ and recalling that $y \in \bigcap\limits_{x\in C} F(x)$, we can deduce that $y \in F(z_t)$ or equivalently, $\sup\limits_{u_t\in T(z_t)} \langle u_t, y - z_t\rangle + \varphi(y) \leq \varphi(z_t)$, $\forall t \in]0,1[$.
Using the convexity of φ, it implies:

$$\sup_{u_t\in T(z_t)} \langle u_t, y - z_t\rangle \leq t(\varphi(z) - \varphi(y)) \quad \forall t \in]0,1[\,.$$

By the convexity of the function $y \to \sup\limits_{u\in T(z_t)} \langle u, y - z_t\rangle$, it follows that

$$0 = \sup_{u_t\in T(z_t)} \langle u_t, z_t - z_t\rangle \leq (1-t) \sup_{u_t\in T(z_t)} \langle u_t, y - z_t\rangle + t \sup_{u_t\in T(z_t)} \langle u_t, z - z_t\rangle.$$

Consequently

$$0 \leq t(1-t)\left(\varphi(z) - \varphi(y)\right) + t \sup_{u_t\in T(z_t)} \langle u_t, z - z_t\rangle,$$

or

$$0 \leq t(1-t)\left(\varphi(z)-\varphi(y)+\sup_{u_t\in T(z_t)}\langle u_t, z-y\rangle\right).$$

Let us first simplify by $t(1-t)$ and let t tend to 0, then from Assumption (d), it follows that,

$$0 \leq \varphi(z)-\varphi(y)+\sup_{v\in T(y)}\langle v, z-y\rangle$$

and the theorem is proved. ∎

Remark that together with the monotonicity of the correspondence T, Assumption (c) of Corollary 3.1 in [GH] implies assumption (c) of the previous theorem. Then, Corollary 3.1 of [GH] is an immediate corollary of the previous theorem.

References

[B] N. Bourbaki, *General Topology: Elements of Mathematics*, Chapters 1-4, (1989) Springer.

[BC1] C. Bardaro and R. Ceppitelli, *Some further generalizations of Knaster-Kuratowski-Mazurkiewicz Theorem and Minimax Inequalities*, J. Math. Anal. Appl. **132** (1989), 484-490.

[BC2] C. Bardaro and R. Ceppitelli, *Fixed point theorems and vector valued minimax theorems*, J. Math. Anal. Appl. **146** (1990), 363-373.

[BCF] H. Ben-El-Mechaiekh, S. Chebbi and M. Florenzano, *A generalized KKMF principle*, J. Math. Anal. Appl. **309** (2005), 583-590.

[C1] B. Cornet, *Fixed point and surjectivity theorems for correspondences; applications*, Ronéotypé Université de Paris-Dauphine (1975).

[C2] B. Cornet, *Thèse de doctorat d'état : Contributions à la théorie mathématique des mécanismes dynamiques d'allocation des ressources*, Université Paris 9 Dauphine, (1981).

[CGH] S. Chebbi, P. Gourdel and H. Hammami, *A Generalization of Fan's Matching Theorem*, Cahiers de la Maison des Sciences Économiques **60** (2006).

[DG] J. Dugundji and A. Granas, *KKM Maps and Variational Inequalities*, (1977).

[DT] X.P. Ding and K.K. Tan, *On equilibria of non compact generalized games*, J. Math. Anal. Appl. **177** (1993), 226-238.

[F] M. Florenzano, *General Equilibrium Analysis Existence and Optimality Properties of Equilibria*, Kluwer Academic Publishers. Boston/Dordrecht/London (2003).

[F1] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305-310.

[F2] K. Fan, *Some properties of convex sets related to fixed point theorems*, Math. Ann. **266** (1984), 519-537.

[GH] P. Gourdel et H. Hammami, *Applications of A Generalized Ky Fan's Matching Theorem in Minimax and Variational Inequality*, (2007).

[HS] P. Hartmann and G. Stampacchia, *On some nonlinear elliptic differential equations*, Acta Math. **115** (1966), 271-310.

[KKM] D. Knaster, C. Kuratowski and S. Mazurkiewicz, *Ein Beweis des Fixpunktsatsez für n-dimensionale simplexe*, Fundamental Mathematics **XIV** (1929), 132-137.

[L] M. Lassonde, *On the Use of KKM correspondences in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151-201.

[Y] C.L. Yen, *A minimax inequality and its applications to variational inequalities*, Pacific Journal Of Mathematics, **Vol 97, No 2** (1981), 477-481.